Consolidated Week 5 Notes. MAST30026 Sep 12, 2014

Lecture 1:

*Def^n: A topological space X is **Hausdorff** if X satisfies
if $\vec{x} = (x_1, x_2, \ldots)$ is a sequence in X and $\lim_{n\to\infty} x_n = q$ and $\lim_{n\to\infty} x_n = p$
then $p = q$.

Note: "limit points are unique"

Note: All metric spaces are Hausdorff.

*Def^n: A topological space X is **compact** if limits exist.
ie- If $\vec{x} = (x_1, x_2, \ldots)$ is a sequence in X then there exists $p \in X$
with $\lim_{n\to\infty} x_n = p$.

- Example: Let $X = [0,1]$ and let $\vec{x} = (x_1, x_2, \ldots)$ be given by

$$x_n = \begin{cases} 1/n, & \text{if } n \text{ is odd} \\ 1 - 1/n, & \text{if } n \text{ is even} \end{cases}$$

$(x_1, x_3, x_5, \ldots)$ is a subsequence converging to 0.

$(x_2, x_4, x_6, \ldots)$ is a subsequence converging to 1.

But, $\lim_{n\to\infty} x_n$ does not exist.

*Def^n: A sequence $\vec{x} = (x_1, x_2, \ldots)$ has a **cluster point** if there exists a subsequence $(x_{n_1}, x_{n_2}, x_{n_3}, \ldots)$ with $n_1 < n_2 < n_3 < \cdots$ which converges.

*Def^n: Let (X, d) be a metric space. The metric space is **sequentially compact** if every sequence $\vec{x} = (x_1, x_2, \ldots)$ in X has a cluster point.

Note: "every sequence has a convergent subsequence"

*Def^n: Let (X, τ) be a topological space. The topological space X is **cover compact** if every open cover has a finite subcover.

Note: If $\mathcal{S} \subseteq \tau$ such that $X = \bigcup_{u \in \mathcal{S}} u$
then there exists $N \in \mathbb{Z}_{>0}$ & $S_1, S_2, \ldots, S_n \in \mathcal{S}$
such that $X = S_1 \cup S_2 \cup S_3 \cup \cdots \cup S_n$

*Def^n: Let (X, d) be a metric space. The metric space X is **ball compact**, or **totally bounded**, or **precompact**, if X satisfies
if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ and $p_1, p_2, \ldots, p_N \in X$
such that $X = B(p_1, \varepsilon) \cup B(p_2, \varepsilon) \cup \cdots \cup B(p_N, \varepsilon)$

*Def^n: The metric space X is **Cauchy compact**, or **complete**, if every Cauchy sequence has a cluster point.

* **HW**: Show that a Cauchy sequence has at most one cluster point.

- **Example:** $\mathbb{R}$ is not compact
- **Proposition:** Let X be a topological space or a metric space & $A \subseteq X$.
  A is sequentially / cover / Cauchy / ball compact if A is sequentially / cover / Cauchy / ball compact as a subspace of X.

** **Theorem:** Let (X, d) be a metric space let $A \subseteq X$.

$$\begin{array}{ccccc} A \text{ is sequentially compact} & \Longrightarrow & A \text{ is Cauchy compact} & \Longrightarrow & A \text{ is closed.} \\ \Updownarrow & & & & \\ A \text{ is cover compact} & \Longrightarrow & A \text{ is ball compact} & \Longrightarrow & A \text{ is bounded.} \end{array}$$

** **Theorem:** Let X be a metric space. & $A \subseteq X$.
If A is Cauchy compact & A is ball compact then A is cover compact.

** **Theorem:** Let $A \subseteq \mathbb{R}^n$
If A is closed and A is bounded, then A is cover compact.

* **HW**: Let $A = (0,1) \subseteq \mathbb{R}^n$. Show that A is ball compact but not cover compact.
Note. Consider sequential compactness, there is a sequence that doesn't converge.

* **HW**: Show that $\mathbb{R}$ is Cauchy compact but not cover compact.

* **HW**: Show that $\mathbb{R}$ is Cauchy compact but not bounded (& hence not ball compact).

* **HW**: What is ball compact but not closed?

Lecture 2:

- **Examples:** (1) $(0,1) \subseteq \mathbb{R}$ is ball compact but not cover compact.
  (2) $(0,1) \subseteq \mathbb{R}$ is ball compact but not closed.
  (3) $\mathbb{R}$ is Cauchy compact but not sequentially compact
  (4) $\mathbb{R}$ is Cauchy compact but not bounded.
  (5) Let $X = (0,1)$ with metric $d(x,y) = |y - x|$
  X is closed in X, but X is not Cauchy compact. $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$

* **HW**: Let (X, d) be a complete metric space and let $A \subseteq X$.
Show that if A is closed then A is complete.

* **HW**: Let (X, d) be a metric space & $A \subseteq X$.
Show that if A complete, then A is closed.

* **HW**: Let (X, d) be a cover compact metric space & $A \subseteq X$.
Show that if A is closed then A is cover compact.

* **HW**: Let (X, d) be a metric space & $A \subseteq X$.

Show that if A is cover compact then A is closed.

- **Example:** $\ell^2 = \{\vec{x} = (x_1, x_2, \dots) \mid x_i \in \mathbb{R}$ such that $\|\vec{x}\|_2 < \infty\}$

with $\|\vec{x}\|_2 = \left(\sum_{i \in \mathbb{Z}_{>0}} x_i^2\right)^{1/2}$

Let $e_1 = (1, 0, 0, \dots)$

$e_2 = (0, 1, 0, \dots)$

$e_3 = (0, 0, 1, 0, \dots)$

Note: $\|e_i\|_2 = (1^2)^{1/2} = 1 < \infty$

So $e_i \in \ell^2$

Let $A = \{e_1, e_2, e_3, \dots\}$. Then $d(e_i, e_j) = \|e_i - e_j\|_2$

$= \|(0, 0, \dots, \underset{i^{th}\text{ position}}{0, 1, 0}, \dots, \underset{j^{th}\text{ position}}{0, -1, 0}, \dots)\|_2$

$= \sqrt{1^2 + (-1)^2}$

$= \sqrt{2}$

So A is bounded.

But, A is not ball compact since there is no way to cover A with a finite number of balls of radius $1/10$.

* HW: Let $A \subseteq \mathbb{R}^n$ $(n \in \mathbb{Z}_{>0})$. Show that if A is bounded then A is ball compact.

* HW: Let $A \subseteq \mathbb{R}^n$ $(n \in \mathbb{Z}_{>0})$. If A is closed then A is complete (Cauchy compact).

Recall: Let (X, d) be a metric space & let $A \subseteq X$.

$$A \text{ is cover compact} \Leftarrow \begin{cases} A \text{ is ball compact} \\ \& \\ A \text{ is Cauchy compact} \end{cases}$$

- **Corollary:** If $A \subseteq \mathbb{R}^n$ & A is closed & A is bounded then A is cover compact.

- **Proposition:** Let X be a topological space & $A \subseteq X$.
Let $f: X \to Y$ be a continuous function.

(a) If A is connected, then $f(A)$ is connected.

(b) If A is cover compact, then $f(A)$ is cover compact.

**** Theorem:** Let $A \subseteq \mathbb{R}$

(a) A is connected if and only if A is an interval

(b) A is connected & compact if and only if $A = [a, b]$ for some $a, b \in \mathbb{R}$ with $a < b$.

**** Intermediate Value Theorem:** If $f: [a, b] \to \mathbb{R}$ is continuous then f attains a a maximum & a minimum & every value in between

ie. $f([a, b]) = [m, M]$

Note: "f(compact connected) is compact connected."

** Rolle's Theorem: If $f:[a,b] \to \mathbb{R}$ is differentiable & $f(a)=f(b)$ then there exists $c \in [a,b]$ such that $f'(c)=0$.

Lecture 3:

** Theorem: Let (X,d) be a metric space & $A \subseteq X$.

$$\begin{array}{ccccc} A \text{ is cover compact} & \Longrightarrow & A \text{ is ball compact} & \Longrightarrow & A \text{ is bounded} \\ \Updownarrow & & & & \\ A \text{ is sequentially compact} & \Longrightarrow & A \text{ is Cauchy compact} & \Longrightarrow & A \text{ is closed} \end{array}$$

* HW: Let $X=\mathbb{R}$ with metric given by $d(x,y) = \min\{|x-y|, 1\}$. Show that X is bounded, but not ball compact.

** Theorem: Let $f: X \to Y$ be a continuous function. Let $E \subseteq X$.

(a) If E is connected then $f(E)$ is connected.

(b) If E is compact then $f(E)$ is compact.

* Def^{n}: A topological space X is locally compact if X is Hausdorff & satisfies if $x \in X$ then there exists a neighborhood N of x (there exists a $U \in \tau$ with $x \in U \subseteq N$) such that N is compact.

* HW: Show that $\mathbb{R}$ is locally compact but not compact.

* Def^{n}: Let X be a locally compact topological space. The one point compactification of X is $X' = X \cup \{\omega\}$ with

$$\tau' = \tau \cup \{(X \setminus K) \cup \{\omega\} \mid K \subseteq X \text{ \& } K \text{ is compact}\}.$$

* HW: Show that τ' is a topology on X'.

* HW: Show that (X', τ') is compact.

- Example: $\mathbb{R}' = \mathbb{R} \cup \{\omega\}$ So, the sequence $1, 2, 3, \ldots$ has cluster point ω.

** Theorem: For topological spaces, cover compact is equivalent to the condition

If $\mathcal{C}$ is a collection of closed sets such that $\bigcap_{C \in \mathcal{C}} C = \emptyset$ then there exists $N \in \mathbb{Z}_{>0}$ and $C_1, C_2, \ldots, C_N \in \mathcal{C}$ such that $C_1 \cap C_2 \cap \cdots \cap C_N = \emptyset$.

* Def^{n}: A topological space (X, τ) is Hausdorff if X satisfies

if $x_1, x_2 \in X$ & $x_1 \neq x_2$ then there exist neighborhoods N_1 & N_2 of x_1 and x_2 ($x_1 \in U_1 \overset{\text{open}}{\subseteq} N_1$ & $x_2 \in U_2 \overset{\text{open}}{\subseteq} N_2$) such that $N_1 \cap N_2 = \emptyset$.

* HW: Show that a metric space is Hausdorff.

Note: Hausdorff is the condition that makes limits unique.

Theorem: Let (X,τ) be a Hausdorff topological space & $A \subseteq X$.
If A is cover compact then A is closed.

HW: Let X be a set with more than one point.
Let $\tau = \{\phi, X\}$. Let $A \subseteq X$ with $A \neq X$, and $A \neq \phi$. Show that A is compact, but not closed.

Def^{n}: A topological space (X,τ) is normal if X satisfies
If C_1 & C_2 are closed subsets of X & $C_1 \cap C_2 = \phi$ then there exist neighborhoods N_1 & N_2 of C_1 & C_2 $(C_1 \subseteq U_1 \overset{open}{\subseteq} N_1,\ C_2 \subseteq U_2 \overset{open}{\subseteq} N_2)$ with $N_1 \cap N_2 = \phi$.

HW: Show that if (X,τ) is a cover compact Hausdorff topological space then X is normal.

Def^{n}: Let (X,τ) be a topological space. The topological space X is path connected if X satisfies:
If $p,q \in X$ then there exists a continuous function $f: [0,1] \to X$ with $f(0) = p$ and $f(1) = q$

HW: Show that if X is path connected then X is connected.
Note: Use "if E is connected then $f(E)$ is connected."

- **Example:** Let X be the graph of $f:[0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

So X is a sub (topological) space of $\mathbb{R}^2$
X is not path connected.
There is no pair of non-empty sets U & V such that $U \cup V \supseteq X$ & $U \cap V = \phi$.
So X is connected.
So connected $\not\Rightarrow$ path connected.

Def^{n}: Let (X,d) be a metric space. A contraction mapping is a function $f: X \to X$ such that there exists $\alpha \in (0,1)$ such that
if $x,y \in X$ then $d(f(x), f(y)) \leq \alpha\, d(x,y)$

Def^{n}: A fixed point of f is $p \in X$ such that $f(p) = p$.

Theorem: (Banach fixed point theorem) Let (X,d) be a complete metric space and $f: X \to X$ a contraction mapping.
(a) There exists a unique fixed point p of f.
(b) Let $x \in X$. Let $x_1 = f(x)$, $x_2 = f(x_1) = f(f(x)) = f^2(x)$, let $x_3 = f^3(x)$
Then $x_1, x_2, x_3, \cdots$ is a convergent sequence in X & $\lim_{n\to\infty} x_n = p$